tively uniform yield of particles over the entire surface of the film. When $Re_e > 4$, the yield of particles shifts in the direction of the edge of the cylinder base. With very large values for Re_e the yield of particles will be noted primarily at the edge of the base, provided that Re_ω is not large. If Re_e and Re_ω are of the same order of magnitude, with both quantities rather large, the particle yield will follow the 3/16 law, i.e.,

$$\frac{\partial C}{\partial \xi}(\eta, h(\eta)) = AC_c\eta^{3/16}. \tag{32}$$

NOTATION

r, φ, z, cylindrical coordinates; V_r, V_φ, V_z, velocity components; P, pressure; E_z, H_φ, components of the electric and magnetic field strengths; ω, angular velocity; T, temperature; h, film thickness; C, concentration; ρ, density of the liquid; ν, kinematic viscosity; g, acceleration of the force of gravity; σ, conductivity; j, current density; μ, magnetic permeability; c_p, specific heat capacity; *a*, coefficient of thermal diffusivity; D, diffusion coefficient.

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TWO-MODEL ITERATION METHOD FOR THE SOLUTION OF AN INVERSE BOUNDARY HEAT-EXCHANGE PROBLEM

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A method is proposed for restoration of the boundary condition, with an iterative correction of the initial data used in this method, involving the utilization of both exact and approximate heat-transfer models.

In constructing installations the design and regime parameters are greatly affected by the projection of heat conditions, this latter significantly dependent on the reliability of processing experimental information as well as on the quality of the mathematical models employed. One of the latest trends in the theory and practice of heat research is based on the application of the principles of inverse heat-exchange problems. The development of effective regularization methods [1, 2] has made it possible to overcome the errors in the formulations of the inverse heat-exchange problems, as had already been noted in [3]. The principle of iteration regularization in extremum formulation of the solution for the inverse heat-exchange problem as proposed by Alifanov and introduction into our examination of the Tikhonov stabilizing functional makes it possible to avoid fluctuations in the restoration parameters in the presence of random errors in the initial temperatures and in errors of approximating differential equations with difference equations.

Formulations of inverse problems, just as in the formulation of direct problems, presuppose the representation of the real heat-exchange process in some mathematical form to express significant factors and interrelationships in the phenomenon being studied. Contemporary methodology of modeling rejects the concept of a "single process — single model." As is validly noted in [4], an entire family of models may be used for any given phenomenon, with these models differing, in particular, in the number of factors which they take into

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consideration and, correspondingly, by the completeness and accuracy with which they describe the process, on the one hand, and by the complexity of the model on the other hand. However, in actual practice such a multimodel situation occurs only in the initial stages of the formulation of the inverse heat-exchange problem, when the researcher, restricted by external limitations and, occasionally, by internal factors, selects a mathematical model in an attempt to satisfy certain, frequently contradictory, requirements such as, for example, adequacy, physical nature, simplicity of realization, limited computer time, etc. In any event, regardless of the manner in which the model was chosen (by guesswork, or by carrying out some test calculations), we find that in the initial stage of developing an algorithm for the solution of the inverse heat-exchange problem we are dealing with a "single-model" situation in which the working model frequently represents such a compromise of requirements which cannot satisfy the researcher, yet no alternative exists.

Thus it would be more correct to deal with the formulation of the inverse problem by connecting measuring devices into the body [4], since these cause distortion in the temperature field in the vicinity where they are positioned, and this distortion may be significant, despite the utilization in these experiments of special devices and methods intended to reduce measurement error. In this case, the researcher faces a choice: either to employ a heat-conduction model which makes allowance for the perturbing effect of the sensor [5, 6], which is cumbersome from the calculation standpoint and requires considerable expenditure of computer time, or he can use a simplified model of a point heat sensor, sacrificing thereby restoration accuracy [7].

This leads us to the idea of extending the multimodel principle to the development stage for the algorithm to solve the inverse heat-exchange problem and thus to avoid compromising the requirements, as well as simultaneously to attempt to satisfy these requirements. In a more narrowly defined version this problem can be reduced to developing a solution method such that, on the one hand, it would guarantee calculation accuracy corresponding to the adequate model of the process and, on the other hand, to achieve an orderly level of complexity in developing the algorithm, as well as to control the cumbersome calculations and expenditure of computer time, characteristic in the utilization of the approximation model.

Essentially, this method, which we will call the two-model method, involves the following. Let there be two models, exact A and approximate $\tilde{A}$, of the heat-transfer process, linking the same causative characteristics u and their consequent effects f:

$$Au = f, \tag{1}$$

$$\tilde{A}u = \tilde{f}, \tag{2}$$

where in the general case $\tilde{f} \neq f$.

Because model A is an adequate representation of the real process, we will assume f to be an initial function in the formulation of the inverse heat-exchange problem:

$$u = A^{-1}f. \tag{3}$$

Let us replace the formulation (3) of the inverse heat-exchange problem with another, in which the operator of the approximate model is used in the place of the exact-model operator, and in this case the unknown function $\tilde{f}$ must be taken as the initial information:

$$u = \tilde{A}^{-1}\tilde{f}. \tag{4}$$

In order for solution (4) to coincide with solution (3), the function $\tilde{f}$ must be related to the function f in terms of (1) and (2). This relationship can be expressed as follows:

$$\tilde{f} = f + \tilde{A}u - Au. \tag{5}$$

Substitution of (5) into (4) gives us the equation for the sought parameter u:

$$u = \tilde{A}^{-1}(f + \tilde{A}u - Au). \tag{6}$$

As is well known, such an equation is effectively solved by means of the iteration method [8]:

$$\overset{(k+1)}{u} = \tilde{A}^{-1}(f + \tilde{A}\overset{(k)}{u} - A\overset{(k)}{u}),\ k = 0,\ 1,\ \ldots \tag{7}$$

We see that there is no operator A^{-1} in (7), i.e., the inverse problem is solved for the approximate model in each iteration of this procedure and, moreover, the solution of the direct problems is found for both the exact and approximate models. If the iteration sequence (7) reduces to a single solution, regardless of the initial approximation, it will serve as a solution of Eq. (6), which means that it serves also as a solution for (3). In actual fact, algorithm (7) by means of iteration serves to correct the initial information, changing the function f into the function $\tilde{f}$. Since this correction comes about through the addition to f of some correction factor, algorithm (7) may logically be referred to as an additive two-model algorithm.

The additive two-model algorithm is iterational, and the question therefore arises as to its convergence and effectiveness.

Convergence. First of all, it should be noted that (7) does not converge for any set $(A, \tilde{A})$. Thus, if we assume $u = u(\tau)$, $f = f(\tau)$, and if we take the integration operator with respect to τ as the operator A, and the operator of multiplication by a fixed number τ_0 as the operator $\tilde{A}$, i.e.,

$$f(\tau)=\int_0^{\tau} u(\tau')\,d\tau',\quad \tilde{f}(\tau)=u(\tau)\,\tau_0,$$

the inverse problem (3) will effectively be a differential function $f(\tau)$; however, as shown by calculations, even for smooth functions $f(\tau)$, the additive two-model algorithm does not yield convergence to the exact solution. Thus, the operators A and $\tilde{A}$ must be subject to certain relationships which would ensure the convergence of the iterations.

We know that when for the solution of the nonlinear equation $x = \varphi(x)$ we use the iteration method $\overset{(k+1)}{x}=\varphi(\overset{(k)}{x})$, $k=0, 1, \ldots$, a sufficient condition for the convergence of the iteration process to a single solution is compressibility of φ [8, 9], i.e., whether for any x_1, $x_2 \in X$ and $0 \le \delta < 1$ we have $\|\varphi(x_1)-\varphi(x_2)\|\le\delta\|x_1-x_2\|$, where $\|\cdot\|$ is the norm in space X.

Consequently, if transformation (6) is compressible, the additive two-model algorithm will provide for convergence to the real solution. Thus, the arbitrary nature of the selection of models A and $\tilde{A}$ is present only within the scope of the limitation which ensures compressibility of (6).

It should be noted that the general requirement for the compressibility of (6), in the actual formulation of the inverse heat-exchange problem, must assume a more specific mathematical form in order to make possible verification of its realization. The need to seek out such limitations on the selection of models A and $\tilde{A}$ from the general condition of compressibility for (6) represents one of the basic problems which arise in the utilization of the two-model algorithm.

Thus, for the linear approximation model $\tilde{A}$ we can obtain the condition

$$\|u_1-u_2-\tilde{A}^{-1}(Au_1-Au_2)\|\le\delta\,\|u_1-u_2\|. \tag{8}$$

When exact model A is also linear, inequality (8) is simplified:

$$\|u-\tilde{A}^{-1}Au\|\le\delta\,\|u\|. \tag{9}$$

Condition (9) imposes a limitation on the selection of linear models A and $\tilde{A}$, which involves the following. If for any given causative function u the direct problem is solved on the basis of the exact model, and if for the initial temperature Au obtained in this manner we then solve the inverse problem through utilization of the approximate model, then the relative deviation in terms of the given norm of the restoration characteristic $\tilde{A}^{-1}Au$ from the original u should not exceed δ. Iteration procedure (7) will then converge and, consequently, (3) has a single solution.

We have to deal separately with the question of how fast the additive two-model algorithm converges. Direct substitution easily demonstrates that regardless of the initial approximation of $\overset{(0)}{u}$, the very first iteration yields an exact solution of the inverse problem (3), if the difference $\tilde{A}u - Au$ is independent of u. Therefore, rapid convergence to an exact solution is to be expected if, in the specific problem, at least approximately a

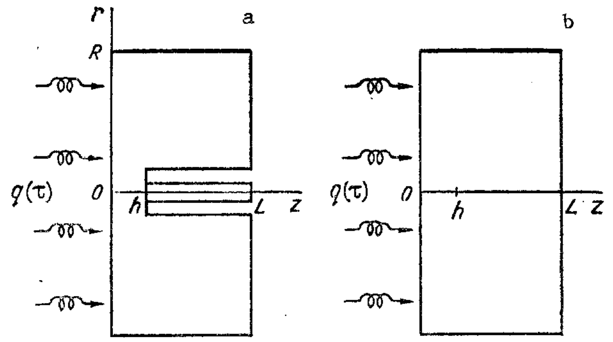

Fig. 1. Regions for the solution of the heat problem in the case of exact (a) and approximate (b) models.

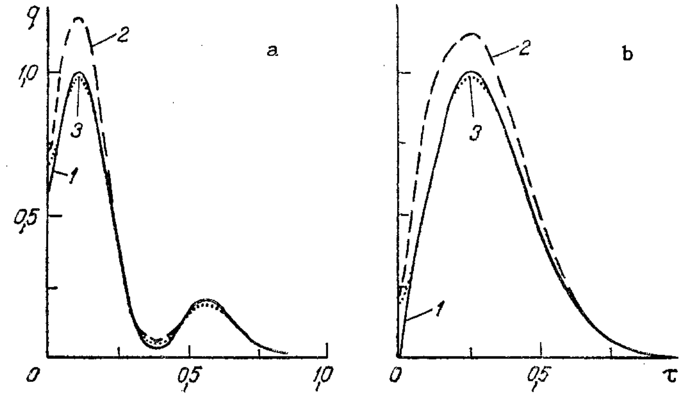

Fig. 2. Restoration of heat-flow density with application of algorithm (7) for unperturbed (a) and perturbed (b) initial data (initially, the zeroth approximation): 1) real solution; 2, 3) 1st and 3rd iterations. q, 10^6 W/m^2; τ, sec.

similar condition is satisfied. It can also be demonstrated that with use of the additive two-model algorithm in the case of linear models A and $\tilde{A}$ and satisfaction of condition (9) we have $\|\overset{(k)}{u}-u^*\|\leq\delta^k\|\overset{(0)}{u}-u^*\|$, where u^* is the exact solution of the inverse problem (3).

Effectiveness. When we use the additive two-model algorithm there is no need for direct solution of the inverse problem for the exact model; therefore, the level of complexity and cumbersome calculations is determined by the development of algorithms for the inverse problem in the case of the approximate model and for the direct problem in the case of the exact model, which assumes a reduction in effort to develop the numerical-calculation algorithm and a savings in computer time.

For example, let the inverse heat-exchange problem through determination of the causative function u (in particular, this may be the boundary density of the heat flow which is exclusively a function of time) be solved in the extremum formulation by the method of conjugate gradients, which calls for n steps of the gradient descent to the minimum of the functional. In consideration of the fact that at each step three problems are solved (the direct problem, the conjugate problem, and in increments), the time required on a computer for the solution of the inverse problem directly on the basis of the exact model amounts to $t_1 = 3nt$, where t is the time expended on the solution of the direct problem on the basis of the exact model. We will also assume that in the solution of the inverse heat-exchange problem by the two-model method the number of iterations in procedure (7) is equal to N, while the time for the solution of the direct problem on the basis of the approximate model amounts to $\tilde{t}$. Then the overall time for the solution of the inverse heat-exchange problem with utilization of the additive two-model algorithm, where the inverse problem for the approximate model is also solved in the extremum formulation, will amount to $t_2 \simeq N(3n\tilde{t} + \tilde{t} + t)$, while the savings in computer time with utilization of the two-model algorithm will be given by the ratio $t_2/t_1 \simeq 3nt/N(3n\tilde{t} + \tilde{t} + t)$. We see that the larger the value of the parameters $t/\tilde{t}$ and n/N, the greater the savings in computer time.

Let us examine an example of realizing the additive two-model algorithm in the solution of the inverse heat-exchange problem through restoration of the density of the heat flow $q(\tau)$, within a cylindrical body of radius R and thickness L. As our exact model we will select the one which takes into consideration the perturbation effect of the heat sensor modeled by the cylinder on the temperature field of the object. In this case, the geometry of the region for the solution of the heat problem will be rather complex (Fig. 1a) and for the calculation of the temperature field we have to use the two-dimensional heat-conduction model:

$$\rho_i c_i \frac{\partial T_i}{\partial \tau} = \lambda_i \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_i}{\partial r} \right) + \frac{\partial^2 T_i}{\partial z^2} \right], \quad 0 < \tau < \tau_m, \; i = 1, 2 \tag{10}$$

$$T_i|_{\tau=0} = 0, \; i = 1, 2, \tag{11}$$

$$-\lambda_1 \frac{\partial T_1}{\partial z}\bigg|_{z=0} = q(\tau), \tag{12}$$

$$\frac{\partial T_1}{\partial r}\bigg|_{r=0} = \frac{\partial T_1}{\partial r}\bigg|_{r=R} = \frac{\partial T_1}{\partial r}\bigg|_{r=r_0, z>h} = \frac{\partial T_2}{\partial r}\bigg|_{r=r_0} = \frac{\partial T_2}{\partial r}\bigg|_{r=r_e} = 0, \tag{13}$$

$$\frac{\partial T_1}{\partial z}\bigg|_{z=h, r_e<r<r_0} = \frac{\partial T_1}{\partial z}\bigg|_{z=L} = \frac{\partial T_2}{\partial z}\bigg|_{z=L} = 0, \tag{14}$$

$$\left(\lambda_1 \frac{\partial T_1}{\partial z} - \lambda_2 \frac{\partial T_2}{\partial z} \right)\bigg|_{z=h, r\leq r_e} = 0, \tag{15}$$

$$(T_1 - T_2)|_{z=h, r\leq r_e} = 0, \tag{16}$$

where $T_i = T_i(r, z, \tau)$ is the temperature field; r_0 and r_e are the radii of the orifice and thermoelectrode; h is the distance from the electrode weld to the heating surface. The subscripts 1 and 2 denote, respectively, the body and the electrode.

Let us now examine the approximation model which makes no provision for the presence of a sensor in the body (Fig. 1b). In this case, the solution of the direct problem can be presented in the form of the Duhamel integral [10]:

$$T_1(z, \tau) = \frac{a_1}{\lambda_1} \int_0^{\tau} q(\xi) G(z; \tau - \xi) d\xi, \tag{17}$$

where a_1 is the coefficient of thermal diffusivity for the body; $G(z; \tau)$ is the Green's function for the flat plate, one of whose boundaries is thermally insulated, while the other edge is subjected to a flow of heat with a density $q(\tau)$.

The following values were assumed for the parameters in the calculations: $R = L = 10^{-2}$ m, $r_0 = 0.6 \cdot 10^{-3}$ m, $r_e = 0.25 \cdot 10^{-3}$ m, $h = 1.0 \cdot 10^{-3}$ m, $\tau_m = 1$ sec. Steel 45 was used as the material for the body, and chromel used for the thermoelectrode. The properties of these materials have been taken from [11].

The numerical determination of the temperature field on the basis of model (10)-(16) was accomplished with a local one-dimensional method with discrete distribution of the space-time grid $n_r \times n_z \times n_\tau = 25 \times 25 \times 1000$.

We took the solution of the direct heat-conduction problem obtained for the exact model (10)-(16) as the initial data for the inverse heat-exchange problem, with some given relationship between the density of the heat flow and time.

The inverse problem for the approximate model (17) at each iteration (7) was solved in the extremum formulation, and the minimization of the functional was accomplished by the method of conjugate gradients, where the cessation of the "adhesion" of the approximations was taken as the unperturbed initial data, while the perturbation data were taken, on the basis of the normal law governing the initial data with mean-square deviation values σ_T = 5% of f_{max}, from the error criteria [1].

Verification of condition (9) for functions $q(\tau) = 1$, $q(\tau) = \tau^2$, and $q(\tau) = 1/(100\tau + 1)$ showed that in all cases the value of δ did not exceed 0.3 for a norm uniform at the segment $[0, \tau_m]$. Thus, we have a basis for the assumption that condition (9) is satisfied, ensuring compressibility of (6), and, consequently, in the given case we can use the two-model algorithm (7) for the solution of the inverse heat-exchange problem.

The results of the solution of the inverse heat-exchange problem are shown in Fig. 2. The first iteration (7) in zeroth initial approximation of the restored function corresponds to the solution of the inverse heat-exchange problem for the approximate model on the uncorrected initial data: $\overset{(1)}{u}=\tilde{A}^{-1}\dot{f}$. At the same time, neglecting the perturbing effect of the heat sensor may lead to a noticeable systematic error in the restored parameter. The process of iteration on the basis of models (7), rapidly converging, made it possible to reduce this error virtually to zero.

For the inverse heat-exchange problem processed in an ES-1045 comptuer, we have t = 27 min, $\tilde{t}$ = 4 sec, N = 5 (here we have taken into consideration the expenditure of time on verification of condition (9), iterations over the models, and the fact that in the first iteration with $\overset{(0)}{q}=0$ there is no need to solve the direct problem for the exact model), n = 9 for the case of unperturbed initial data, which corresponds to values of t_1 = 141 min and t_2 = 729 min. This indicates an economy of almost ten hours of computer time. For the perturbed initial data the calculation yielded values of t_1 = 138 min and t_2 = 405 min.

In conclusion, it should be noted that the two-model algorithm, just as iteration algorithms in general, are easily achievable on a computer, and all questions relating to inaccuracies and regularization in the solution of inverse heat-exchange problems must be relegated to the approximate model.

NOTATION

τ, current time; r, z, cylindrical coordinates; ρ, density; c, specific heat capacity; λ, thermal conductivity.

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